Automated Solution of Heterogeneous Agent Models

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Problem

Interactions of forward looking heterogeneous agents subject to aggregate disturbances induce a law of motion for distributions. We characterize a tractable subclass admitting an automated, fast, guaranteed accurate approximate solution algorithm generalizing first order DSGE solution methods to allow function-valued states.

- *Model*: $\mathbb{E}F(x, y, x', y', \sigma) = 0$
  - x, y (collections of) functions
- *Solution*: find operators $h(x), g(x)$ s.t. $y = g(x)$ $x' = h(x) + \sigma z'$ satisfy model
- *Goal*: $\tilde{g}_x, \tilde{h}_x$: approximate functional derivatives around $\sigma = 0$ steady state by map from basis coefficients to basis coefficients

Conditions

Represent model equations F as sequence of nodes composed along *Directed Acyclic Graph*. Nodes are functional neural network layers: compose *differentiable pointwise nonlinearities* $f([g(s)])$ and *linear integral operators* $\int k(s_2, s_1)[g(s_1)]ds_1$

Setup allows linearization with standard autodiff: functions only evaluated on grid so differentiation and discretization commute. This may require modified representation of model equations, such as by applying change-of-variables.

Eg: $\int V'(g(X) + u) f_U(u)du$ should be $\int V'(s) f_U(s - g(X))ds$

With regularity conditions on smoothness and graph topology, discretization can be mapped back to function space with vanishing error.

Why it works

With structure and post-processing, linearized model is system of *Fredholm Integral Equations*

$$F_{g(.)}[g()] = g(s') + \int k(s', s)[g(s)]ds$$

$k(.,.)$ is product of derivatives along nodes.

Algorithm approximates basis projection $\tilde{F}_g \approx$

$$I + \sum_{i,j=1}^{K} \langle k(s', s), \phi_i(s')\varphi_j(s)\rangle \langle ., \varphi_j(s)\rangle \phi_i(s')$$

Input vs. Output distinction, post-processing ensure kernel is interpolated, I is identity matrix.

This produces solution as a map on basis function coefficients while computing on grid.

Convergence Guarantees:

Under model form and smoothness conditions can apply *Young's Inequality*:

L^∞−approximation of $k(.,.)$ $\implies$

$$\sup_{\|g\|_{L^2}=1} \left\| (\tilde{F}_g - F_g)[g(.)] \right\|_{L^2} \to 0$$

Childers (2018): this uniform (operator norm) approximation + Blanchard-Kahn (1980)-type eigenvalue conditions $\implies$ DSGE solver output converges uniformly.

Total running time is $O(K^3)$ in # of grid points with error proportional to L^∞ error for $K-$point interpolation of kernel functions $k(.,.)$. Rates depend on interaction of smoothness of primitives and choice of basis.

Generic Model Class

Structure of heterogeneous agent models

1. *Optimization*: Bellman Equation

$$V(X, \epsilon) = \max_{Y=g(X,\epsilon)} u(Y, X, P, \epsilon) + \beta\mathbb{E}$$

$$\int\int V'(X', \epsilon') f_U(U') f_\epsilon(\epsilon', P')dU'd\epsilon'$$

$$\text{st } X' = Q(Y, X, P, P', U')$$

2. *Aggregation*: Kolmogorov Equation

$$f'_X(X') = \int\int f_U(Q^{-1}_{g(X,\epsilon),X}(X'))$$

$$\left|\det \frac{\partial}{dX'} Q^{-1}_{g(X,\epsilon),X}(X')\right| f_X(X) f_\epsilon(\epsilon, P)dXd\epsilon$$

3. *Aggregate Shocks*: (Functional) AR Processes

$$P'_2 = h_P(P_2, \sigma Z')$$

4. *Market Clearing*: $F(f_X(.), g(.), P) = 0$

Policy, value, distribution function, aggregates $(g(.,.), V(.), f_X(.), P)$ are endogenous states

Algorithm

1. Replace function-valued states $(x, y) = \{g_j(.)\}_{j=1}^{p^\ell}$ by values at grid points

$$\vec{g}_j = \{g_j(s_{[j]i})\}_{i=1}^{K_j}$$

2. Replace integral operators by quadrature with abcissas at grid

$$\int k(s_2, s_1) g(s_1) ds_1 \approx \sum_{i=1}^{K_{[s_1]}} \pi_i k(s_{2j}, s_{1i}) g(s_{1i})$$

3. Solve for steady state vectors $\{g_j^*(s_{[j]i})\}_{i=1}^{K_j}$
4. Categorize input $\{\vec{g}_j\}_{j=1}^{2d_2}$, output $\{\vec{g}_{j(p^\ell)}\}_{p^\ell=2d_2+1}^{2d_2+o^\ell}$ versions of arguments
5. Linearize wrt $\{\vec{g}_{p^\ell}\}_{p^\ell=1}^{2d_2+o^\ell}$ at g^* by autodiff
6. Apply post-processing transformations
7. Pre, post multiply input Jacobians by interpolation matrix $M_{[j]}$
   - Maps grid points to basis $\Phi_{[j]}$ coefs
8. Apply standard linear RE solver to interpolated Jacobians

Example: Huggett Model

One bond consumption-savings model with aggregate and iid idiosyncratic income risk and borrowing constraint. Solve for joint law of wealth distribution $m(.)$, consumption function $c(.)$, interest rate R and aggregate income shock z.

$$\ell(w) = \mathbb{E}\beta R \int\int g(w' - R(w + z - c(w + z, \ell(w), R)) -$$
$$s')c(w' + z', \ell'(w'), R')^{-\gamma} ds'dw' \quad (1)$$

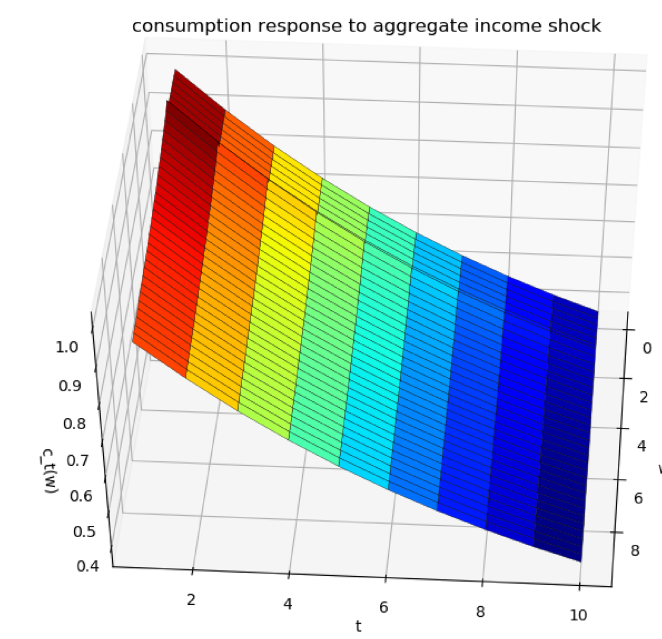
$$\text{where } c(w, \ell, R) := \min\left\{\ell^{-1/\gamma}, w - \frac{a}{R}\right\}$$

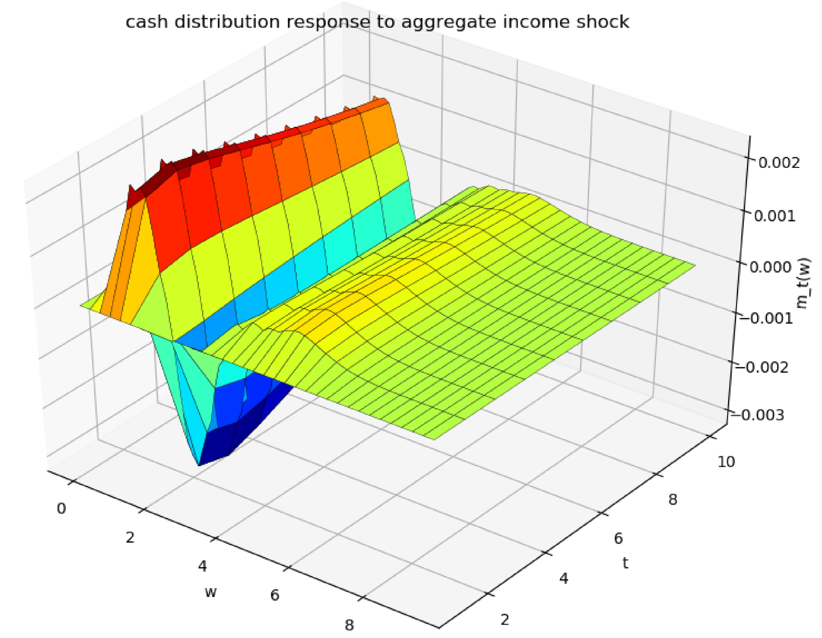
$$m'(w') = \int\int g(w' - R(w + z - c(w + z,$$
$$\ell(w), R)) - s')m(w)ds'dw' \quad (2)$$

$$z' = \rho_z z + \sigma\epsilon' \quad (3)$$

$$\int (w + z - c(w + z, \ell(w), R))m(w)dw = 0 \quad (4)$$

Uses Parameterized Expectations Euler equation in expected marginal utility $\ell()$ of cash-on-hand w to ensure smooth compositional structure.





Impulse Responses of Consumption Function and Wealth Distribution to Aggregate Income shock